Exercises

Integration rules – Solutions

Exercise 1.

(a)
$$\begin{aligned}\int e^{2x}(3x-2)\,dx &= \frac{1}{2}e^{2x}(3x-2) - \int \frac{1}{2}e^{2x}\cdot 3dx + c\\ &= \frac{1}{2}e^{2x}(3x-2) - \frac{3}{2}\int e^{2x}dx + c\\ &= \frac{1}{2}e^{2x}(3x-2) - \frac{3}{2}\cdot\frac{1}{2}e^{2x} + c\\ &= \frac{1}{2}e^{2x}\left(3x-\frac{7}{2}\right) + c\end{aligned}$$

(b)
$$\begin{aligned}\int_0^{\pi}(x+1)\sin(x)\,dx &= [(x+1)(-\cos(x))]_0^{\pi} - \int_0^{\pi}(-\cos(x))\,dx\\ &= (\pi+1)-(-1)-[-\sin(x)]_0^{\pi}\\ &= \pi+2-0=\pi+2\end{aligned}$$

(c)
$$\begin{aligned}\int(2x^2-3)\cdot\sin(x) &= -\cos(x)(2x^2-3) - \int -\cos(x)4x\,dx + c\\ &= -\cos(x)(2x^2-3) + \int \cos(x)4x\,dx + c\\ &= -\cos(x)(2x^2-3) + \sin(x)4x - \int \sin(x)\cdot 4dx + c\\ &= -\cos(x)(2x^2-3) + \sin(x)4x + 4\cos(x) + c\end{aligned}$$

Exercise 2. (a) We will use the substitution $t := (2x+3)$. Then $\frac{dt}{dx} = 2$ implying that $dx = \frac{dt}{2}$.

$$\begin{aligned}\int_{-1}^{0}\frac{1}{(2x+3)^4}dx &= \int_1^3 \frac{1}{t^4}\frac{dt}{2}\\ &= \frac{1}{2}\left[-\frac{1}{3}\frac{1}{t^3}\right]_1^3\\ &= \frac{1}{6}\cdot\left(-\frac{1}{3^3}+\frac{1}{(-1)^3}\right)\\ &= \frac{1}{6}\cdot\left(\frac{26}{27}\right) = \frac{13}{81}\end{aligned}$$

(b) We will use the substitution $t := -x^2 + 1$. Then $\frac{dt}{dx} = -2x$ implying that $dx = \frac{dt}{-2x}$.

$$\begin{aligned}\int xe^{-x^2+1}dx &= \int xe^t \frac{dt}{-2x} \\ &= -\frac{1}{2}\int e^t dt \\ &= -\frac{1}{2}e^t + c = -\frac{1}{2}e^{-x^2+1} + c\end{aligned}$$

(c) We will use the substitution $t := x^2$. Then $\frac{dt}{dx} = 2x$ implying that $dx = \frac{dt}{2x}$.

$$\begin{aligned}\int x\cos(x^2)dx &= \int x\cos(t)\frac{dt}{2x} \\ &= \frac{1}{2}\int \cos(t)dt \\ &= \frac{1}{2}\sin(x) + c = \frac{1}{2}\sin(x^2) + c\end{aligned}$$

(d) We will use the substitution $t := \sin(x)$. Then $\frac{dt}{dx} = \cos(x)$ implying that $dx = \frac{dt}{\cos(x)}$.

$$\begin{aligned}\int \frac{\cos(x)}{\sin(x)}dx &= \int \frac{\cos(x)}{t}\frac{dt}{\cos(x)} \\ &= \int \frac{1}{t}dt \\ &= \ln(|t|) + c = \ln(|\sin(x)|) + c\end{aligned}$$

(e) We will use the substitution $t := -x^3 + 1$. Then $\frac{dt}{dx} = -3x^2$ implying that $dx = \frac{dt}{-3x^2}$.

$$\begin{aligned}\int x^2e^{-x^3+1}dx &= \int x^2e^t\frac{dt}{-3x^2} \\ &= -\frac{1}{3}\int e^t dt \\ &= -\frac{1}{3}e^t + c = -\frac{1}{3}e^{-x^3+1} + c\end{aligned}$$

(f) We will use the substitution $t := \ln(x)$. Then $\frac{dt}{dx} = \frac{1}{x}$ implying that $dx = x \cdot dt$.

$$\begin{aligned} \int_1^e \frac{\sqrt{\ln(x)}}{x} dx &= \int_0^1 \frac{\sqrt{t}}{x} \cdot x dt \\ &= \int_0^1 \sqrt{t} dt \\ &= \left[\frac{2}{3} t^{\frac{3}{2}} dt \right]_0^1 = \frac{2}{3} \cdot 1 - \frac{2}{3} \cdot 0 = \frac{2}{3} \end{aligned}$$

Exercise 3. We compute with partial integration:

$$\begin{aligned} \int e^x \cos(x) dx &= e^x \cos(x) - \int e^x (-\sin(x)) dx + c \\ &= e^x \cos(x) + \int e^x \sin(x) dx + c \\ &= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x) dx + c \end{aligned}$$

If we set $A := \int e^x \cos(x) dx$ then we get

$$A = e^x \cos(x) + e^x \sin(x) - A$$

which is equivalent to

$$A = \frac{1}{2} \left(e^x \cos(x) + e^x \sin(x) \right).$$

Exercise 4.

1.

$$
\begin{aligned}
\int_{-\infty}^{\infty} t \cdot f(t) dt &= \int_{0}^{\infty} \lambda t e^{-\lambda t} dt \\
&= \lim_{\alpha \to \infty} \lambda \int_{0}^{\alpha} t e^{-\lambda t} dt \\
&= \lim_{\alpha \to \infty} \lambda \left(\left[t \cdot \left(-\frac{1}{\lambda} e^{-\lambda t} \right) \right]_{0}^{\alpha} - \int_{0}^{\alpha} -\frac{1}{\lambda} e^{-\lambda t} dt \right) \\
&= \lim_{\alpha \to \infty} \lambda \left(-\alpha \cdot \frac{1}{\lambda} e^{-\lambda \alpha} + \left[-\frac{1}{\lambda} e^{-\lambda t} \right]_{0}^{\alpha} \right) \\
&= \lim_{\alpha \to \infty} -\alpha e^{-\lambda \alpha} - \frac{1}{\lambda} \underbrace{\lim_{\alpha \to \infty} e^{-\lambda \alpha}}_{=0} + \frac{1}{\lambda} \\
&= - \lim_{\alpha \to \infty} \frac{\alpha}{e^{\lambda \alpha}} + \frac{1}{\lambda} \\
&\stackrel{L'H}{=} - \lim_{\alpha \to \infty} \frac{1}{-\lambda e^{\lambda \alpha}} + \frac{1}{\lambda} \\
&= 0 + \frac{1}{\lambda} = \frac{1}{\lambda}
\end{aligned}
$$

2. We will use the substitution rule with the substitution $z := -\frac{t^2}{2}$. Then $\frac{dz}{dt} = -t$ and thus $dt = \frac{dz}{-t}$

$$
\begin{aligned}
\int_{-\infty}^{\infty} t \cdot f(t) dt &= \int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} t e^{-\frac{t^2}{2}} dt \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to \infty} \int_{\frac{\alpha^2}{2}}^{\frac{\alpha^2}{2}} t e^{z} \frac{dz}{-t} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to \infty} \int_{-\frac{\alpha^2}{2}}^{-\frac{\alpha^2}{2}} -e^{z} dz \\
&= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to \infty} \left[-e^{z} \right]_{-\frac{\alpha^2}{2}}^{-\frac{\alpha^2}{2}} = 0
\end{aligned}
$$